

Bounded Solutions and Topological Linearization of DEPCAGs with Unbounded Nonlinear Terms

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Abstract

The existence, uniqueness and stability of differential equations with piecewise constant argument (DEPCAs) and the differential equation with piecewise constant argument of generalized type (DEPCAGs) has been well investigated recently. Seldom did the authors study the linearization problem of such systems except [28] and [30]. However, they studied the linearization problem based on that the nonlinear terms in the systems are bounded. In this paper, under the assumption that the nonlinear term is unbounded, we study the bounded solution and global topological linearization of a class of DEPCAGs. A new criterion is given for the existence of a unique bounded solution. Moreover, some sufficient conditions are established for the topological conjugacy between a nonlinear system and its linear system. some novel techniques are employed. The main results in previous papers are improved.

Keywords: differential equation; fundamental matrix; quaternion; solution; noncommutativity; eigenvalue

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1 Introduction and Motivation

In recent years, there has been an increasing interest in studying differential equations with piecewise constant argument of generalized type (DEPCAGs). It takes the form of

$$z'(t) = M(t)z(t) + M_0(t)z(\gamma(t)) + h(t, z(t), z(\gamma(t))), \quad (1.1)$$

where $t \in \mathbb{R}$, $z(t) \in \mathbb{R}^n$, $M(t)$ and $M_0(t)$ are $n \times n$ matrices, $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\gamma(t) : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following condition **(A)**:

There exist two constant sequences $\{t_i\}_{i \in \mathbb{Z}}$ and $\{\zeta_i\}_{i \in \mathbb{Z}}$ such that

(A1) $t_i < t_{i+1}$ and $t_i \leq \zeta_i \leq t_{i+1}$, $\forall i \in \mathbb{Z}$,

(A2) $t_i \rightarrow \pm\infty$ as $i \rightarrow \pm\infty$,

(A3) $\gamma(t) = \zeta_i$ for $t \in [t_i, t_{i+1})$,

(A4) there exists a constant $\theta > 0$ such that $t_{i+1} - t_i \leq \theta$, $\forall i \in \mathbb{Z}$.

In particular, when $\gamma(t) = [t]$ or $\gamma(t) = 2[\frac{t+1}{2}]$, system (1.1) is called the differential equations with piecewise constant argument (DEPCAs).

For DEPCAs and DEPCAGs, many scholars study the existence, uniqueness, continuity, boundedness and stability of the solutions. One can refer to [1, 3, 16, 13, 11, 38, 32]. In particular, the bounded solutions of DEPCAs and DEPCAGs were obtained in [29, 2, 12, 33, 15]. Among these works, Akhmet [2] obtained a set of sufficient conditions to guarantee the existence of a unique bounded solution by assuming that the linear system $z'(t) = M(t)z(t)$

in system (1.1) has an exponential dichotomy. But if $M(t) = 0$, then $z'(t) = M(t)z(t)$ can not admit an exponential dichotomy. In this case, the result in [2] is invalid. Later, Akhmet [5, 6] introduced the condition that the linear system with piecewise constant argument

$$z'(t) = M(t)z(t) + M_0(t)z(\gamma(t)) \quad (1.2)$$

admits an exponential dichotomy. Under the assumption that linear system (1.2) admits an exponential dichotomy and the nonlinear term $h(t, z(t), z(\gamma(t)))$ is bounded, he proved that there exists a unique bounded solution of system (1.1) (see [15, Th. 5.3]). What happens if the nonlinear term $h(t, z(t), z(\gamma(t)))$ is unbounded? Does there exist a unique bounded solution? This paper is devoted to answering this question. We prove that even if $h(t, z(t), z(\gamma(t)))$ is unbounded, system (1.1) has a unique bounded solution under some suitable conditions. We briefly summarize our result on bounded solution as follows:

Result 1 *Assume that system (1.2) admits an exponential dichotomy and the nonlinear term $h(t, z(t), z(\gamma(t)))$ is Lipschitzian. If we further assume that there exist constant numbers $r > 0$ and $\mu > 0$, such that*

$$|h(t, z(t), z(\gamma(t)))| \leq r(|z(t)| + |z(\gamma(t))|) + \mu,$$

then system (1.1) has a unique bounded solution under some conditions.

Remark 1 We point out that $h(t, z(t), z(\gamma(t)))$ can be a polynomial of order one about $z(t)$ and $z(\gamma(t))$, which can be unbounded. Thus our result improves Theorem 5.3 in [15].

Another purpose in this paper is to apply Theorem 1 to study the linearization of system (1.1) when the nonlinear term $h(t, z(t), z(\gamma(t)))$ is unbounded. Topological linearization is one of the most important subjects of differential equations. A brief survey about topological linearization is presented as follows:

The classical linearization theorem was given for the autonomous differential equations by Hartman and Grobman [17, 19]. Later, Palmer generalized the Hartmann-Grobman theorem to the nonautonomous case in [25, 26]. For the ordinary differential equations, Shi and Xiong [37], Shi [36], Jiang [20, 21], Barreira and Valls [7, 8] extended Palmer's result in various directions. For example, Shi and Xiong [37] reduced the conditions by assuming that the partial linear subsystem admits an exponential dichotomy; Jiang [20, 21] reduced the condition by assuming that the linear system admits a generalized dichotomy; Barreira and Valls [7, 8] reduced the condition by assuming that the linear system admits a nonuniform exponential dichotomy. In addition, topological linearization of difference equations and functional differential equations have been extensively studied. For example, see [10, 22, 27, 23, 40, 34, 41, 42, 43, 44].

However, seldom did the authors study the linearization problem of DEPCAGs. In 1996, Papaschinopoulos [28] generalized the topological linearization theorem to DEPCAs. And nineteen years later, Pinto and Robledo [33] generalized the work of Papaschinopoulos to DEPCAGs. Under suitable conditions, they proved that the above nonlinear system (1.1) is topologically conjugated to its linear system (1.2). They studied the linearization problem

based on that the nonlinear terms in the systems are bounded. More specifically, the results in [28] and [33] require that $h(t, z(t), z(\gamma(t)))$ is bounded, ie., there exists a constant $\mu > 0$ such that

$$|h(t, z(t), z(\gamma(t)))| \leq \mu.$$

However, in general, $h(t, z(t), z(\gamma(t)))$ can be unbounded. For example, taking $h(t, z(t), z(\gamma(t))) = z(t)$ or $h(t, z(t), z(\gamma(t))) = z(\gamma(t))$. In this case, the results in [28] and [33] are not valid. In this paper, we prove that if $h(t, z(t), z(\gamma(t)))$ is unbounded, system (1.1) can also be topologically conjugated to system (1.2) as long as it has a proper structure. More precisely, we consider the following system

$$\begin{cases} x'(t) = A(t)x(t) + A_0(t)x(\gamma(t)) + f(t, x(t), x(\gamma(t))) + \phi(t, y(t), y(\gamma(t))), \\ y'(t) = B(t)y(t) + B_0(t)y(\gamma(t)) + g(t, x(t), x(\gamma(t))) + \psi(t, y(t), y(\gamma(t))), \end{cases} \quad (1.3)$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^{n_1}$, $y(t) \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, $A(t)$, $A_0(t)$ are $n_1 \times n_1$ matrices, $B(t)$, $B_0(t)$ are $n_2 \times n_2$ matrices, $f : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_1}$, $g : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$, $\phi : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_1}$, and $\psi : \mathbb{R} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_2}$.

In this paper, under the assumption that the nonlinear term is unbounded, we study the global topological linearization of a class of DEPCAGs (1.3) based on Result 1. Some novel techniques are employed in the proof. We briefly summarize our second result on bounded solution as follows:

Result 2 *Suppose that the linear system*

$$\begin{cases} x'(t) = A(t)x(t) + A_0(t)x(\gamma(t)), \\ y'(t) = B(t)y(t) + B_0(t)y(\gamma(t)), \end{cases} \quad (1.4)$$

admits an exponential dichotomy. Assume that the nonlinear terms $f(t, x(t), x(\gamma(t)))$, $g(t, x(t), x(\gamma(t)))$ are Lipschitzian. If we further assume that there exist constant λ and $\delta > 0$ such that

$$|f(t, x(t), x(\gamma(t)))| \leq \lambda(|x(t)| + |x(\gamma(t))|), \quad |g(t, x(t), x(\gamma(t)))| \leq \lambda(|x(t)| + |x(\gamma(t))|),$$

$$|\phi(t, y(t), y(\gamma(t)))| \leq \delta, \quad |\psi(t, y(t), y(\gamma(t)))| \leq \delta.$$

Then system (1.3) is topologically conjugated to system (1.4) under proper conditions.

Remark 2 As you will see, the nonlinear terms $f(t, x(t), x(\gamma(t)))$ and $g(t, x(t), x(\gamma(t)))$ can be possibly unbounded. For example, $f(t, x(t), x(\gamma(t)))$ and $g(t, x(t), x(\gamma(t)))$ can be polynomials of order one about $x(t)$. In this case, the nonlinear term of system (1.4) is unbounded, however the topological linearization can be realized. This result improves some results in [28] and [33].

Remark 3 Some novel techniques are employed to prove our main result on linearization. Due to the unboundedness of the nonlinear terms, it is difficult to directly prove that the nonlinear system (1.3) is topologically conjugated to the linear system (1.4). To overcome such difficulty, we should introduce the auxiliary system as follows

$$\begin{cases} x'(t) = A(t)x(t) + A_0(t)x(\gamma(t)) + f(t, x(t), x(\gamma(t))), \\ y'(t) = B(t)y(t) + B_0(t)y(\gamma(t)) + g(t, x(t), x(\gamma(t))). \end{cases} \quad (1.5)$$

We first prove that system (1.4) is topologically conjugated to system (1.5). Secondly, we prove that system (1.5) and system (1.3) are topologically conjugated. Then, by transition of topological conjugacy, system (1.4) and system (1.3) are topologically conjugated.

The rest of this paper is organized as follows: In Section 2, we give some definitions, notation and preliminary lemmas. Our main results, Theorem 1 and Theorem 2, are stated in Section 3. The proof of Theorem 1 is given in Section 4. The proof of Theorem 2 is very long and we divide the proofs into several Sections (see Sections 5-8).

2 Preliminaries

2.1 General assumptions

We introduce two groups of assumptions for Theorem 1 and Theorem 2, respectively. The first group is conditions **(B, C)** and the second group is conditions **(B, C)**. In this paper, $|\cdot|$ denotes a vector norm or matrix norm.

We assume that system (1.1) and (1.2) satisfy the following conditions.

Condition **(B)**:

(B1) The functions $M(t)$, $M_0(t)$ and $h(t, z(t), z(\gamma(t)))$ are locally integrable in \mathbb{R} .

(B2) There exists constants $r > 0$, $\mu > 0$ and $\ell > 0$ such that for any $t \in \mathbb{R}$, $(t, z(t), z(\gamma(t)))$ and $(t, \hat{z}(t), \hat{z}(\gamma(t))) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$,

$$|h(t, z(t), z(\gamma(t)))| \leq r(|z(t)| + |z(\gamma(t))|) + \mu,$$

and

$$|h(t, z(t), z(\gamma(t))) - h(t, \hat{z}(t), \hat{z}(\gamma(t)))| \leq \ell(|z(t) - \hat{z}(t)| + |z(\gamma(t)) - \hat{z}(\gamma(t))|).$$

We remark that if we further assume that $r \leq \ell$, $|h(t, 0, 0)| \leq \mu$, the Lipschitz condition in (B2) implies the first estimation $|h(t, z, y)| \leq r(|z| + |y|) + \mu$ in (B2).

Moreover, we introduce the following notation and condition **(C)**.

(i) We define $I_i = [t_i, t_{i+1})$ for any $i \in \mathbb{Z}$.

(ii) For any $i \in \mathbb{Z}$ and $k \times k$ matrix $Q(t)$, we define

$$\rho_i^+(Q) = \exp\left(\int_{t_i}^{\zeta_i} |Q(s)| ds\right) \quad \text{and} \quad \rho_i^-(Q) = \exp\left(\int_{\zeta_i}^{t_{i+1}} |Q(s)| ds\right).$$

Condition **(C)**: There exists $0 < \nu^+ < 1$ and $0 < \nu^- < 1$ such that the matrices $M(t)$ and $M_0(t)$ satisfy properties:

$$\sup_{i \in \mathbb{Z}} \rho_i^+(M) \ln \rho_i^+(M_0) \leq \nu^+, \quad \sup_{i \in \mathbb{Z}} \rho_i^-(M) \ln \rho_i^-(M_0) \leq \nu^-,$$

and

$$1 \leq \rho(M) \triangleq \sup_{i \in \mathbb{Z}} \rho_i^+(M) \rho_i^-(M) < +\infty. \quad (2.1)$$

Therefore,

$$\rho_0(M) \triangleq \rho(M)^2 \left(\frac{1+\nu^-}{1-\nu^+}\right) > 1. \quad (2.2)$$

Now, we introduce conditions **(B, C)** for systems (1.3) and (1.4).

Condition **(B)**:

(B1) There exist constants $\beta > 0$ and $\beta_0 > 0$ such that

$$\begin{aligned} \sup_{t \in \mathbb{R}} |A(t)| &\leq \beta, & \sup_{t \in \mathbb{R}} |B(t)| &\leq \beta, \\ \sup_{t \in \mathbb{R}} |A_0(t)| &\leq \beta_0, & \sup_{t \in \mathbb{R}} |B_0(t)| &\leq \beta_0. \end{aligned}$$

(B2) There exist constants $\delta > 0$ and $\lambda > 0$ such that for any $(t, x(t), x(\gamma(t))) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1}$ and $(t, y(t), y(\gamma(t))) \in \mathbb{R} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$,

$$\begin{aligned} |f(t, x(t), x(\gamma(t)))| &\leq \lambda(|x(t)| + |x(\gamma(t))|), \\ |g(t, x(t), x(\gamma(t)))| &\leq \lambda(|x(t)| + |x(\gamma(t))|), \\ |\phi(t, y(t), y(\gamma(t)))| &\leq \delta, \\ |\psi(t, y(t), y(\gamma(t)))| &\leq \delta. \end{aligned}$$

(B3) There exists constant $\omega > 0$ such that for any $(t, x_1(t), x_1(\gamma(t))), (t, x_2(t), x_2(\gamma(t))) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1}$ and $(t, y_1(t), y_1(\gamma(t))), (t, y_2(t), y_2(\gamma(t))) \in \mathbb{R} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$,

$$\begin{aligned} &|f(t, x_1(t), x_1(\gamma(t))) - f(t, x_2(t), x_2(\gamma(t)))| \\ &\leq \omega \left(|x_1(t) - x_2(t)| + |x_1(\gamma(t)) - x_2(\gamma(t))| \right), \\ &|g(t, x_1(t), x_1(\gamma(t))) - g(t, x_2(t), x_2(\gamma(t)))| \\ &\leq \omega \left(|x_1(t) - x_2(t)| + |x_1(\gamma(t)) - x_2(\gamma(t))| \right), \end{aligned}$$

$$\begin{aligned}
& |\phi(t, y_1(t), y_1(\gamma(t))) - \phi(t, y_2(t), y_2(\gamma(t)))| \\
& \leq \omega(|y_1(t) - y_2(t)| + |y_1(\gamma(t)) - y_2(\gamma(t))|),
\end{aligned}$$

and

$$\begin{aligned}
& |\psi(t, y_1(t), y_1(\gamma(t))) - \psi(t, y_2(t), y_2(\gamma(t)))| \\
& \leq \omega(|y_1(t) - y_2(t)| + |y_1(\gamma(t)) - y_2(\gamma(t))|).
\end{aligned}$$

Condition **(C)**: There exist $0 < \nu^+ < 1$ and $0 < \nu^- < 1$ such that matrices $A(t)$, $A_0(t)$, $B(t)$ and $B_0(t)$ satisfy following properties:

$$\begin{aligned}
\sup_{i \in \mathbb{Z}} \rho_i^+(A) \ln \rho_i^+(A_0) &\leq \nu^+, & \sup_{i \in \mathbb{Z}} \rho_i^-(A) \ln \rho_i^-(A_0) &\leq \nu^-, \\
\sup_{i \in \mathbb{Z}} \rho_i^+(B) \ln \rho_i^+(B_0) &\leq \nu^+, & \sup_{i \in \mathbb{Z}} \rho_i^-(B) \ln \rho_i^-(B_0) &\leq \nu^-.
\end{aligned}$$

Note that **(B1)** and **(A4)** imply that

$$1 \leq \rho(A) \triangleq \sup_{i \in \mathbb{Z}} \rho_i^+(A) \rho_i^-(A) < +\infty \quad \text{and} \quad 1 \leq \rho(B) \triangleq \sup_{i \in \mathbb{Z}} \rho_i^+(B) \rho_i^-(B) < +\infty. \quad (2.3)$$

Thus,

$$\rho_0(A) \triangleq \rho^2(A) \left(\frac{1+\nu^-}{1-\nu^+} \right) > 1 \quad \text{and} \quad \rho_0(B) \triangleq \rho^2(B) \left(\frac{1+\nu^-}{1-\nu^+} \right) > 1. \quad (2.4)$$

Throughout the rest of the paper, we assume that conditions **(A, B, C, B, C)** hold.

2.2 Notation of solutions for DEPCAGs

The notion of solutions for DEPCAGs was introduced in [1, 6, 12, 14, 15, 39].

Definition 2 (Solutions of a DEPCAG) A continuous function $z(t)$ is a solution of system (1.1) or system (1.2) on \mathbb{R} if:

- (i) The derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of points $t_i, i \in \mathbb{Z}$, where the one side derivative exists;
- (ii) The equation is satisfied for $z(t)$ on each interval (t_i, t_{i+1}) and it holds for the right derivative of $z(t)$ at t_i .

2.3 Transition matrices

In this subsection, we introduce some notation associated with solutions of a class of DEPCAGs.

Let $\Phi(t)$ be the fundamental matrix of system $x' = M(t)x$ with $\Phi(0) = I$. For any $t \in I_j$, $\tau \in I_i$, $s \in \mathbb{R}$, we introduce the following notations adopting from [15, 31, 33]:

$$\Phi(t, s) = \Phi(t)\Phi^{-1}(s),$$

$$J(t, \tau) = I + \int_{\tau}^t \Phi(\tau, s)M_0(s)ds,$$

$$E(t, \tau) = \Phi(t, \tau) + \int_{\tau}^t \Phi(t, s)M_0(s)ds = \Phi(t, \tau)J(t, \tau).$$

We define backward and forward products of a set of $k \times k$ matrices $\mathcal{Q}_i (i = 1, \dots, m)$ as follows:

$$\prod_{i=1}^{\leftarrow m} \mathcal{Q}_i = \begin{cases} \mathcal{Q}_m \cdots \mathcal{Q}_2 \mathcal{Q}_1, & \text{if } m \geq 1, \\ I, & \text{if } m < 1, \end{cases}$$

and

$$\prod_{i=1}^{\rightarrow m} \mathcal{Q}_i = \begin{cases} \mathcal{Q}_1 \mathcal{Q}_2 \cdots \mathcal{Q}_m, & \text{if } m \geq 1, \\ I, & \text{if } m < 1. \end{cases}$$

If $J(t, s)$ is nonsingular, we could define the transition matrix $Z(t, s)$ of system (1.2) as follows:
if $t > \tau$,

$$\begin{aligned} & Z(t, \tau) \\ &= E(t, \zeta_j)E(t_j, \zeta_j)^{-1} \prod_{r=i+2}^{\leftarrow j} \left(E(t_r, \gamma(t_{r-1}))E(t_{r-1}, \gamma(t_{r-1}))^{-1} \right) E(t_{i+1}, \gamma(\tau))E(\tau, \gamma(\tau))^{-1}, \end{aligned}$$

if $t < \tau$,

$$\begin{aligned} & Z(t, \tau) \\ &= E(t, \zeta_j)E(t_{j+1}, \zeta_j)^{-1} \prod_{r=j+1}^{\rightarrow i-1} \left(E(t_r, \gamma(t_r))E(t_{r+1}, \gamma(t_r))^{-1} \right) E(t_i, \gamma(\tau))E(\tau, \gamma(\tau))^{-1}. \end{aligned}$$

Through simple calculations, we obtain $Z(t, \tau)Z(\tau, s) = Z(t, s)$ and $Z(t, s) = Z(s, t)^{-1}$. Since $E(\tau, \tau) = I$ and $\frac{\partial E(t, \tau)}{\partial t} = M(t)E(t, \tau) + M_0(t)$, we have

$$\frac{\partial Z(t, \tau)}{\partial t} = M(t)Z(t, \tau) + M_0(t)Z(\gamma(t), \tau).$$

Thus, $Z(t, \tau)$ is a solution of system (1.2).

2.4 Formulas of solutions for DEPCAGs

To introduce the formulas of solutions, we first state the following important lemma.

Lemma 2.1 ([31], Lemma 4.3) *Assume that conditions $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ are fulfilled, then $J(t, s)$ is nonsingular for any $t, s \in \bar{I}_r$ and the matrices $Z(t, s)$ and $Z(t, s)^{-1}$ are well defined for any $t, s \in \mathbb{R}$. If $t, s \in \bar{I}_r$, then*

$$\begin{aligned} |\Phi(t, s)| &\leq \rho(M), \\ |Z(t, s)| &\leq \rho_0(M), \end{aligned}$$

where $\rho(\cdot)$ is defined in (2.1) and $\rho_0(\cdot)$ is defined in (2.2).

We remark that Lemma 2.1 ensures the continuity of solutions of system (1.1) on \mathbb{R} . We introduce the following formulas for DEPCAGs.

Proposition 2.1 ([31], p.239) *For any $t \in I_j$, $\tau \in I_i$, the solution of system (1.2) with $x(\tau) = \xi$ is defined on \mathbb{R} and is given by*

$$z(t) = Z(t, \tau)\xi. \quad (2.5)$$

Proposition 2.2 ([31], Th 3.3) *For any $t \in I_j$, $\tau \in I_i$ and $t > \tau$, the solution of system (1.1) with $z(\tau) = \xi$ is defined on \mathbb{R} and is given by*

$$\begin{aligned} z(t) = & Z(t, \tau)\xi + \int_{\tau}^{\zeta_i} Z(t, \tau)\Phi(\tau, s)h(s)ds + \sum_{r=i+1}^j \int_{t_r}^{\zeta_r} Z(t, t_r)\Phi(t_r, s)h(s)ds \\ & + \sum_{r=i}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z(t, t_{r+1})\Phi(t_{r+1}, s)h(s)ds + \int_{\zeta_j}^t \Phi(t, s)h(s)ds, \end{aligned} \quad (2.6)$$

where $h(s) = h(s, z(s), z(\gamma(s)))$.

Remark 2.1 If $t < \tau$, one could obtain the solution formula by replacing $\sum_{r=i+1}^j$ and $\sum_{r=i}^{j-1}$ by

$\sum_{r=j+1}^i$ and $\sum_{r=j}^{i-1}$, respectively.

2.5 Subsystems of System (1.3)

For convenience, consider the following subsystems of system (1.3):

$$x'(t) = A(t)x(t) + A_0(t)x(\gamma(t)) + f(t, x(t), x(\gamma(t))) + \phi(t, y(t), y(\gamma(t))), \quad (2.7)$$

$$y'(t) = B(t)y(t) + B_0(t)y(\gamma(t)) + g(t, x(t), x(\gamma(t))) + \psi(t, y(t), y(\gamma(t))), \quad (2.8)$$

and subsystems of system (1.4):

$$x'(t) = A(t)x(t) + A_0(t)x(\gamma(t)), \quad (2.9)$$

$$y'(t) = B(t)y(t) + B_0(t)y(\gamma(t)). \quad (2.10)$$

Let $\Phi_1(t)$ be the fundamental matrix of system $x' = A(t)x$ with $\Phi_1(0) = I$, and $\Phi_2(t)$ be the fundamental matrix of system $y' = B(t)y$ with $\Phi_2(0) = I$.

For any $t \in I_j$, $\tau \in I_i$, $s \in \mathbb{R}$, similar to $\Phi(t, s)$, $J(t, \tau)$, and $E(t, \tau)$ in subsection 2.3, we could define

$$\Phi_k(t, s), \quad J_k(t, \tau) \quad \text{and} \quad E_k(t, \tau), \quad k = 1, 2.$$

If $J_k(t, s)$ ($k = 1, 2$) is nonsingular, we could define the transition matrices $Z_1(t, s)$ and $Z_2(t, s)$ of subsystems (2.9) and (2.10), respectively. Moreover, we could verify that $Z_1(t, \tau)$ and $Z_2(t, \tau)$ are solutions of subsystems (2.9) and (2.10), respectively.

2.6 α -exponential dichotomy and Green function

Now we introduce the definition of exponential dichotomy for a DEPCAG. In this paper, we adopt the following definition from Akhmet [5, 6].

Definition 3 (α -exponential dichotomy for a DEPCAG) The linear system (1.2) has an α -exponential dichotomy on \mathbb{R} if there exist a projection P , constants $K \geq 1$ and $\alpha > 0$ such that the transition matrix $Z(t, s)$ of system (1.2) satisfies

$$|Z_P(t, s)| \leq K e^{-\alpha|t-s|},$$

where $Z_P(t, s)$ is defined by

$$Z_P(t, s) = \begin{cases} Z(t, 0)PZ(0, s), & t \geq s, \\ -Z(t, 0)(I - P)Z(0, s), & s > t. \end{cases}$$

For convenience, we define the Green function corresponding to system (1.1) which was introduced in [33, 15]. Given $t \in (\zeta_j, t_{j+1})$,

$$\tilde{G}(t, s) = \begin{cases} Z_p(t, t_r)\Phi(t_r, s), & \text{if } s \in [t_r, \zeta_r) \text{ for any } r \in \mathbb{Z}, \\ Z_p(t, t_{r+1})\Phi(t_{r+1}, s) & \text{if } s \in [\zeta_r, t_{r+1}) \text{ for any } r \in \mathbb{Z} \setminus \{j\}, \\ \Phi(t, s) & \text{if } s \in [\zeta_j, t), \\ 0 & \text{if } s \in [t, t_{j+1}], \end{cases}$$

and if $t \in [t_j, \zeta_j]$,

$$\tilde{G}(t, s) = \begin{cases} Z_p(t, t_r)\Phi(t_r, s) & \text{if } s \in [t_r, \zeta_r) \text{ for any } r \in \mathbb{Z} \setminus \{j\}, \\ Z_p(t, t_{r+1})\Phi(t_{r+1}, s) & \text{if } s \in [\zeta_r, t_{r+1}) \text{ for any } r \in \mathbb{Z}, \\ 0 & \text{if } s \in [t_j, t), \\ -\Phi(t, s) & \text{if } s \in [t, \zeta_j), \end{cases}$$

We denote $\tilde{G}_1(t, s) = \tilde{G}(t, s)$ for $t \geq s$ and $\tilde{G}_2(t, s) = -\tilde{G}(t, s)$ for $t < s$.

2.7 Condition (\mathfrak{D})

For convenience, we apply the following condition in our second result to replace the condition that system (1.4) has an α -exponential dichotomy.

Condition (\mathfrak{D}) : There exist constants $K \geq 1$ and $\alpha > 0$ such that

$$|Z_1(t, s)| \leq e^{-\alpha(t-s)}, \quad t \geq s \quad \text{and} \quad |Z_2(t, s)| \leq Ke^{\alpha(t-s)}, \quad s > t.$$

It is clear that condition (\mathfrak{D}) is equivalent to assume that system (1.4) has an α -exponential dichotomy by taking $K = 1$ in the first inequality. We point out that this assumption is natural. In fact, we can get the inequality by taking another equivalent norm or supposing the following conditions:

$$\frac{d|x(t)|'}{dt} \Big|_{(1.4)} \leq -2\alpha|x(t)|^2, \quad |f(t, x(t), x(\gamma(t)))| \leq \frac{\alpha}{2}|x|.$$

2.8 Topological conjugacy

The notion of topological equivalence and topological conjugacy can be found in [25, 26, 33, 42].

Definition 1 (Topological conjugacy) A continuous function $H : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is topological equivalence between system (1.1) and (1.2) if following conditions hold:

- (i) for each $t \in \mathbb{R}$, $H(t, z)$ is a homeomorphism of \mathbb{R}^n ,
- (ii) $H(t, z) - z$ is bounded in $\mathbb{R} \times \mathbb{R}^n$,
- (iii) if $z(t)$ is a solution of system (1.1), then $H(t, z(t))$ is a solution of system (1.2).

In addition, the function $L(t, z) = H^{-1}(t, z)$ has properties (i)-(iii) also.

If such a map H exists, then system (1.1) and (1.2) are called topologically conjugated.

2.9 Some lemmas

Lemma 2.2 ([33], Proposition 3) *If system (1.2) has an α -exponential dichotomy on \mathbb{R} , then \tilde{G} satisfies*

$$|\tilde{G}(t, s)| \leq K\rho^*(M)e^{-\alpha|t-s|},$$

where $\rho^*(M) = \rho(M)e^{\alpha\theta}$, $\rho(M)$ is defined in (2.1) and θ is in (A4).

From Lemma 2.2, we have that

$$|\tilde{G}_1(t, s)| \leq K\rho^*(M)e^{-\alpha(t-s)} \quad \text{for } t \geq s, \quad |\tilde{G}_2(t, s)| \leq K\rho^*(M)e^{-\alpha(s-t)} \quad \text{for } t < s. \quad (2.11)$$

Lemma 2.3 ([33], Lemma 2.3) *If system (1.2) has an α -exponential dichotomy on \mathbb{R} , then the unique solution bounded on \mathbb{R} is the null solution.*

3 Main Results

Now we are in a position to state our main results.

Theorem 1 *If conditions (A, B, C) hold and system (1.2) has an α -exponential dichotomy with constant $K \geq 1$ and $\alpha > 0$, further assume that*

$$8Kl\rho^*(M)\alpha^{-1} \leq 1, \quad 4Kr\rho^*(M)\alpha^{-1} \leq 1, \quad (3.1)$$

where $\rho^*(M)$ is defined in Lemma 2.2, then system (1.1) has a unique solution bounded on \mathbb{R} which can be represented as follows

$$z(t) = \int_{-\infty}^t \tilde{G}_1(t, s)h(s, z(s), z(\gamma(s)))ds - \int_t^{+\infty} \tilde{G}_2(t, s)h(s, z(s), z(\gamma(s)))ds,$$

and

$$|z(t)| \leq 2K\mu\tilde{\rho}(M)(\alpha - 4rK\tilde{\rho}(M))^{-1} \triangleq \sigma.$$

We remark that if system (1.1) reduces to ODE, that is,

$$z'(t) = M(t)z(t) + h(t, z(t), \xi),$$

Theorem 1 is valid for ODE.

Theorem 2 *If conditions $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ hold, further assume that*

$$8K\tilde{\rho}(A)\omega\alpha^{-1} < 1, \quad 8K\tilde{\rho}(B)\omega\alpha^{-1} < 1, \quad (3.2)$$

$$16K\tilde{\rho}(A)\lambda\alpha^{-1} < 1, \quad 16K\tilde{\rho}(B)\lambda\alpha^{-1} < 1, \quad (3.3)$$

$$\alpha_0 = \alpha - 2\omega\tilde{\rho}(A)e^{\alpha\theta} > 0, \quad (3.4)$$

$$F(\ell, \theta)(\beta_0 + \ell)\theta = v < 1, \quad (3.5)$$

where $F(\ell, \theta) = \frac{e^{(\beta+\ell)\theta}-1}{(\beta+\ell)\theta}$, $\tilde{\rho}(\cdot) = \max(\rho(\cdot)\rho_0(\cdot), \rho(\cdot)e^{\alpha\theta})$, $\rho(\cdot)$ is defined in (2.3), $\rho_0(\cdot)$ is defined in (2.4), then system (1.3) is topologically conjugated to system (1.4).

4 The proof of Theorem 1

To prove Theorem 1, we first introduce the following lemma.

Lemma 4.1 *If $t > \zeta_i$ and $z(t)$ is a bounded solution of system (1.1), then*

$$I \triangleq \sum_{r=-\infty}^i \int_{t_r}^{\zeta_r} Z(t, 0)PZ(0, t_r)\Phi(t_r, s)h(s, z(s), z(\gamma(s)))ds$$

is convergent.

Proof From $t_r \leq \zeta_r$, $t > \zeta_i$, (B) and (2.11), we have

$$\begin{aligned} |I| &\leq \int_{-\infty}^t |Z(t, 0)PZ(0, t_r)\Phi(t_r, s)h(s, z(s), z(\gamma(s)))|ds \\ &\leq \int_{-\infty}^t |\tilde{G}_1(t, s)|r(|z(s)| + |z(\gamma(s))|) + \mu ds \\ &\leq \int_{-\infty}^t K\rho^*(M)e^{-\alpha(t-s)}(2r|z| + \mu)ds \\ &= K\rho^*(M)\alpha^{-1}(2r|z| + \mu). \end{aligned}$$

Since $z(s)$ is a bounded solution, I is convergent. \square

The proof of Theorem 1:

Proof For σ defined in Theorem 1, denote

$$\Omega = \{\varphi(t) | \varphi : \mathbb{R} \rightarrow \mathbb{R}^n \text{ is continuous and } |\varphi(t)| \leq \sigma\},$$

and

$$W = \{\varphi(t) | \varphi : \mathbb{R} \rightarrow \mathbb{R}^n \text{ is continuous and } \|\varphi\| < \infty\}.$$

It is easy to see that W is a Banach space and Ω is a closed subset of W .

Suppose that $t \in [\zeta_j, t_{j+1})$, $0 \in [t_i, \zeta_i]$, $j > i$. For any $\varphi(t) \in \Omega$, define the map $T : \Omega \rightarrow W$ as follows

$$\begin{aligned} T\varphi(t) &= \sum_{r=-\infty}^j \int_{t_r}^{\zeta_r} Z(t, 0) P Z(0, t_r) \Phi(t_r, s) h(s, \varphi(s), \varphi(\gamma(s))) ds \\ &+ \sum_{r=-\infty}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z(t, 0) P Z(0, t_{r+1}) \Phi(t_{r+1}, s) h(s, \varphi(s), \varphi(\gamma(s))) ds \\ &- \sum_{r=j+1}^{+\infty} \int_{t_r}^{\zeta_r} Z(t, 0) (I - P) Z(0, t_r) \Phi(t_r, s) h(s, \varphi(s), \varphi(\gamma(s))) ds \\ &- \sum_{r=j}^{+\infty} \int_{\zeta_r}^{t_{r+1}} Z(t, 0) (I - P) Z(0, t_{r+1}) \Phi(t_{r+1}, s) h(s, \varphi(s), \varphi(\gamma(s))) ds \\ &+ \int_{\zeta_j}^t \Phi(t, s) h(s, \varphi(s), \varphi(\gamma(s))) ds \\ &= \int_{-\infty}^t \tilde{G}_1(t, s) h(s, \varphi(s), \varphi(\gamma(s))) ds - \int_t^{+\infty} \tilde{G}_2(t, s) h(s, \varphi(s), \varphi(\gamma(s))) ds. \end{aligned}$$

To prove the existence and uniqueness of bounded solution, we divide it into two steps.

Step 1 We prove that the map T has a unique fixed point by contraction principle.

Due to (2.11) and (??), we get

$$\begin{aligned} |T\varphi(t)| &\leq \int_{-\infty}^t K e^{-\alpha(t-s)} \tilde{\rho}(M) (r|z(t)| + r|z(\gamma(t))| + \mu) ds \\ &+ \int_t^{+\infty} K e^{\alpha(t-s)} \tilde{\rho}(M) (r|z(t)| + r|z(\gamma(t))| + \mu) ds \\ &\leq [K \tilde{\rho}(M) (\mu + 2r\sigma) + K \tilde{\rho}(M) (\mu + 2r\sigma)] \alpha^{-1} \\ &= 2K \tilde{\rho}(M) \alpha^{-1} (\mu + 2r\sigma) \\ &= \sigma. \end{aligned}$$

Therefore $T\varphi \in \Omega$ and T is a map from Ω to Ω .

For any $\varphi_1(t), \varphi_2(t) \in \Omega$, from (2.11) and (B2) we have

$$\begin{aligned}
|T\varphi_1(t) - T\varphi_2(t)| &= \left| \int_{-\infty}^t \tilde{G}_1(t, s)[h(s, \varphi_1(s), \varphi_1(\gamma(s))) - h(s, \varphi_2(s), \varphi_2(\gamma(s)))]ds \right. \\
&\quad \left. + \int_t^{+\infty} \tilde{G}_2(t, s)[h(s, \varphi_1(s), \varphi_1(\gamma(s))) - h(s, \varphi_2(s), \varphi_2(\gamma(s)))]ds \right| \\
&\leq \int_{-\infty}^t K\tilde{\rho}(M)e^{-\alpha(t-s)}l(|\varphi_1(s) - \varphi_2(s)| + |\varphi_1(\gamma(s)) - \varphi_2(\gamma(s))|) \\
&\quad + \int_t^{+\infty} K\tilde{\rho}(M)e^{\alpha(t-s)}l(|\varphi_1(s) - \varphi_2(s)| + |\varphi_1(\gamma(s)) - \varphi_2(\gamma(s))|)ds \\
&\leq 2Kl\tilde{\rho}(M)\alpha^{-1}\|\varphi_1 - \varphi_2\| + 2Kl\tilde{\rho}(M)\alpha^{-1}\|\varphi_1 - \varphi_2\| \\
&\leq \frac{1}{2}\|\varphi_1 - \varphi_2\|.
\end{aligned}$$

Thus T is a contraction map in Ω . By the contraction map principle, there exists a unique $\varphi_0(t) \in \Omega$ such that

$$\begin{aligned}
\varphi_0(t) &= T\varphi_0(t) \\
&= \sum_{r=-\infty}^j \int_{t_r}^{\zeta_r} Z(t, 0)PZ(0, t_r)\Phi(t_r, s)h(s, \varphi_0(s), \varphi_0(\gamma(s)))ds \\
&\quad + \sum_{r=-\infty}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z(t, 0)PZ(0, t_{r+1})\Phi(t_{r+1}, s)h(s, \varphi_0(s), \varphi_0(\gamma(s)))ds \\
&\quad - \sum_{r=j+1}^{+\infty} \int_{t_r}^{\zeta_r} Z(t, 0)(I - P)Z(0, t_r)\Phi(t_r, s)h(s, \varphi_0(s), \varphi_0(\gamma(s)))ds \\
&\quad - \sum_{r=j}^{+\infty} \int_{\zeta_r}^{t_{r+1}} Z(t, 0)(I - P)Z(0, t_{r+1})\Phi(t_{r+1}, s)h(s, \varphi_0(s), \varphi_0(\gamma(s)))ds \\
&\quad + \int_{\zeta_j}^t \Phi(t, s)h(s, \varphi_0(s), \varphi_0(\gamma(s)))ds.
\end{aligned}$$

Furthermore, it is easy to check that $\varphi_0(t)$ is a solution of system (1.1).

Step 2 We prove the uniqueness of the bounded solution. That is, we prove that $\varphi_0(t)$ is the unique bounded solution of system (1.1). In fact, suppose that $\varphi_1(t)$ is another bounded

solution of system (1.1), by Proposition 2.2, we get

$$\begin{aligned}
\varphi_1(t) &= Z(t, 0)\varphi_1(0) + \int_0^{\zeta_i} Z(t, 0)\Phi(0, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds \\
&\quad + \sum_{r=i+1}^j \int_{t_r}^{\zeta_r} Z(t, t_r)\Phi(t_r, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds \\
&\quad + \sum_{r=i}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z(t, t_{r+1})\Phi(t_{r+1}, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds \\
&\quad + \int_{\zeta_j}^t \Phi(t, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds \\
&= Z(t, 0)\{\varphi_1(0) + \int_0^{\zeta_i} \Phi(0, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds \\
&\quad + \sum_{r=i+1}^j P \int_{t_r}^{\zeta_r} Z(0, t_r)\Phi(t_r, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds \\
&\quad + \sum_{r=i}^{j-1} P \int_{\zeta_r}^{t_{r+1}} Z(0, t_{r+1})\Phi(t_{r+1}, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds\} \\
&\quad + \int_{\zeta_j}^t \Phi(t, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds \\
&\quad + Z(t, 0)\{\sum_{r=i+1}^j (I - P) \int_{t_r}^{\zeta_r} Z(0, t_r)\Phi(t_r, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds \\
&\quad + \sum_{r=i}^{j-1} (I - P) \int_{\zeta_r}^{t_{r+1}} Z(0, t_{r+1})\Phi(t_{r+1}, s)h(s, \varphi_1(s), \varphi_1(\gamma(s)))ds\}
\end{aligned}$$

By Lemma 4.1, we have that

$$\begin{aligned}
\varphi_1(t) &= Z(t, 0)(\varphi_1(0) + c_0) \\
&+ \sum_{r=-\infty}^j \int_{t_r}^{\zeta_r} Z(t, 0) P Z(0, t_r) \Phi(t_r, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds \\
&+ \sum_{r=-\infty}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z(t, 0) P Z(0, t_{r+1}) \Phi(t_{r+1}, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds \\
&- \sum_{r=j+1}^{+\infty} \int_{t_r}^{\zeta_r} Z(t, 0) (I - P) Z(0, t_r) \Phi(t_r, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds \\
&- \sum_{r=j}^{+\infty} \int_{\zeta_r}^{t_{r+1}} Z(t, 0) (I - P) Z(0, t_{r+1}) \Phi(t_{r+1}, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds \\
&+ \int_{\zeta_j}^t \Phi(t, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds \\
&\triangleq Z(t, 0)(\varphi_1(0) + c_0) + J,
\end{aligned}$$

where

$$\begin{aligned}
c_0 &= \int_0^{\zeta_i} \Phi(0, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds \\
&- \sum_{r=-\infty}^i \int_{t_r}^{\zeta_r} P Z(0, t_r) \Phi(t_r, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds \\
&- \sum_{r=-\infty}^{i-1} \int_{\zeta_r}^{t_{r+1}} P Z(0, t_{r+1}) \Phi(t_{r+1}, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds, \\
&+ \sum_{r=i+1}^{+\infty} \int_{t_r}^{\zeta_r} (I - P) Z(0, t_r) \Phi(t_r, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds \\
&+ \sum_{r=i}^{+\infty} \int_{\zeta_r}^{t_{r+1}} (I - P) Z(0, t_{r+1}) \Phi(t_{r+1}, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds.
\end{aligned}$$

Similar to the computation of $|T\varphi(t)|$, we could prove that J is bounded. Thus $Z(t, 0)(\varphi_1(0) + c_0)$ is a bounded solution of system (1.2). From Lemma 2.3, we have

$$\varphi_1(0) + c_0 = 0.$$

Thus

$$\varphi_1(t) = \int_{-\infty}^t \tilde{G}_1(t, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds - \int_t^{+\infty} \tilde{G}_2(t, s) h(s, \varphi_1(s), \varphi_1(\gamma(s))) ds.$$

Furthermore,

$$\begin{aligned}
& |\varphi_1(t) - \varphi_0(t)| \\
& \leq \left| \int_{-\infty}^t \tilde{G}_1(t, s) [h(s, \varphi_1(s), \varphi_1(\gamma(s))) - h(s, \varphi_0(s), \varphi_0(\gamma(s)))] ds \right| \\
& + \left| \int_t^{+\infty} \tilde{G}_2(t, s) [h(s, \varphi_1(s), \varphi_1(\gamma(s))) - h(s, \varphi_0(s), \varphi_0(\gamma(s)))] ds \right| \\
& \leq \int_{-\infty}^t Kl\tilde{\rho}(M)e^{-\alpha(t-s)}(|\varphi_1(s) - \varphi_0(s)| + |\varphi_1(\gamma(s)) - \varphi_0(\gamma(s))|) ds \\
& + \int_t^{+\infty} Kl\tilde{\rho}(M)e^{\alpha(t-s)}(|\varphi_1(s) - \varphi_0(s)| + |\varphi_1(\gamma(s)) - \varphi_0(\gamma(s))|) ds \\
& \leq 4Kl\tilde{\rho}(M)\alpha^{-1}\|\varphi_1 - \varphi_0\| \\
& \leq \frac{1}{2}\|\varphi_1 - \varphi_0\|.
\end{aligned}$$

Therefore

$$\|\varphi_1 - \varphi_0\| \leq \frac{1}{2}\|\varphi_1 - \varphi_0\|,$$

which implies that $\varphi_1(t) = \varphi_0(t)$. This completes the proof. \square

5 The preliminaries for the proof of Theorem 2

In this section, we give some preliminaries for the proof of Theorem 2.

5.1 The solutions of subsystems

From Lemma 2.1, we have the following lemma.

Lemma 5.1 *Assume that conditions $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ are fulfilled, then $J_k(t, s)$ ($k = 1, 2$) is nonsingular for any $t, s \in \bar{I}_r$ and the matrices $Z_k(t, s)$ and $Z_k(t, s)^{-1}$ ($k = 1, 2$) are well defined for any $t, s \in \mathbb{R}$. If $t, s \in \bar{I}_r$, then*

$$|\Phi_1(t, s)| \leq \rho(A), \quad |\Phi_2(t, s)| \leq \rho(B),$$

$$|Z_1(t, s)| \leq \rho_0(A), \quad |Z_2(t, s)| \leq \rho_0(B),$$

where $\rho(\cdot)$ is defined in (2.3) and $\rho_0(\cdot)$ is defined in (2.4).

Lemma 5.1 ensures the continuity of solutions of subsystems (2.7) and (2.8) on \mathbb{R} . Moreover, we give the following remark.

Remark 5.1 The fundamental matrix $\Phi(t)$ of system $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} A(t)x(t) \\ B(t)y(t) \end{pmatrix}$ with $\Phi(0) = I$, and the transition matrix $Z(t, s)$ of system (1.4) have the following form

$$\Phi(t, s) = \begin{pmatrix} \Phi_1(t, s) & 0 \\ 0 & \Phi_2(t, s) \end{pmatrix}, \quad Z(t, s) = \begin{pmatrix} Z_1(t, s) & 0 \\ 0 & Z_2(t, s) \end{pmatrix}.$$

From Proposition 2.1, for any $t \in I_j$, $\tau \in I_i$, the solution of subsystem (2.9) with $x(\tau) = \xi$ is defined on \mathbb{R} and is given by

$$x(t) = Z_1(t, \tau)\xi, \quad (5.1)$$

and the solution of subsystem (2.10) with $y(\tau) = \eta$ can be represented as

$$y(t) = Z_2(t, \tau)\eta. \quad (5.2)$$

From Proposition 2.2, for any $t \in I_j$, $\tau \in I_i$ and $t > \tau$, the solution of subsystem (2.7) with $x(\tau) = \xi$ is defined on \mathbb{R} and is given by

$$\begin{aligned} x(t) &= Z_1(t, \tau)\xi + \int_{\tau}^{\zeta_i} Z_1(t, \tau)\Phi_1(\tau, s)(f(s) + \phi(s))ds + \sum_{r=i+1}^j \int_{t_r}^{\zeta_r} Z_1(t, t_r)\Phi_1(t_r, s)(f(s) + \phi(s))ds \\ &\quad + \sum_{r=i}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z_1(t, t_{r+1})\Phi_1(t_{r+1}, s)(f(s) + \phi(s))ds + \int_{\zeta_j}^t \Phi_1(t, s)(f(s) + \phi(s))ds \\ &\triangleq Z_1(t, \tau)\xi + \int_{\tau}^t G_1(t, s)(f(s) + \phi(s))ds, \end{aligned} \quad (5.3)$$

where $f(s) = f(s, x(s), x(\gamma(s)))$, $\phi(s) = \phi(s, y(s), y(\gamma(s)))$ and

$$G_1(t, s, \tau) = \begin{cases} Z_1(t, \tau)\Phi_1(\tau, s), & \text{if } s \in [\tau, \zeta_i] \text{ or } s \in [\zeta_i, \tau], \\ Z_1(t, t_r)\Phi_1(t_r, s), & \text{if } s \in [t_r, \zeta_r) \text{ for } r = i+1, \dots, j, \\ Z_1(t, t_{r+1})\Phi_1(t_{r+1}, s) & \text{if } s \in [\zeta_r, t_{r+1}) \text{ for } r = i, \dots, j-1, \\ \Phi_1(t, s) & \text{if } s \in [\zeta_j, t] \text{ or } s \in [t, \zeta_j]. \end{cases}$$

Similarly, if $t > \tau$, the solution of subsystem (2.8) with $y(\tau) = \eta$ can be represented as

$$\begin{aligned} y(t) &= Z_2(t, \tau)\eta + \int_{\tau}^{\zeta_i} Z_2(t, \tau)\Phi_2(\tau, s)(g(s) + \psi(s))ds + \sum_{r=i+1}^j \int_{t_r}^{\zeta_r} Z_2(t, t_r)\Phi_2(t_r, s)(g(s) + \psi(s))ds \\ &\quad + \sum_{r=i}^{j-1} \int_{\zeta_r}^{t_{r+1}} Z_2(t, t_{r+1})\Phi_2(t_{r+1}, s)(g(s) + \psi(s))ds + \int_{\zeta_j}^t \Phi_2(t, s)(g(s) + \psi(s))ds \\ &= Z_2(t, \tau)\eta + \int_{\tau}^t G_2(t, s)(g(s) + \psi(s))ds, \end{aligned} \quad (5.4)$$

where $g(s) = g(s, x(s), x(\gamma(s)))$, $\psi(s) = \psi(s, y(s), y(\gamma(s)))$ and $G_2(t, s)$ can be defined in the same way as $G_1(t, s)$.

Remark 5.2 We could obtain $G_k(t, s)$ ($k = 1, 2$) for $t < \tau$ by replacing $r = i + 1, \dots, j$, and $r = i, \dots, j - 1$, with $r = j + 1, \dots, i$, and $r = j, \dots, i - 1$, in the definitions of $G_k(t, s)$ ($t > s, k = 1, 2$), respectively. From Remark 2.1, one could obtain the solution formulas of subsystems (2.7) and (2.8) for the case $t < \tau$.

5.2 Some lemmas

Lemma 5.2 *If condition (\mathfrak{D}) holds, for $t \in \mathbb{R}$ and $s \in \mathbb{R}$, then*

$$|G_1(t, s)| \leq K\tilde{\rho}(A)e^{-\alpha(t-s)}, \quad t \geq s, \quad |G_2(t, s)| \leq K\tilde{\rho}(B)e^{\alpha(t-s)}, \quad t < s,$$

where $\tilde{\rho}(\cdot)$ is defined in Theorem 1, α is in (\mathfrak{D}) and θ is in **(A4)**.

Proof We just prove the first inequality.

Suppose that $t \in I_j$, $\tau \in I_i$ and $t \geq s$.

Case 1. $t \geq \tau$.

Without loss of generality, we assume that $t_i \leq \tau \leq \zeta_i \leq t_{i+1} \leq \dots \leq t_j \leq \zeta_j \leq t$.

If $s \in [\tau, \zeta_i]$, due to **(A4)**, we have $s - \tau \leq \theta$. It follows from (\mathfrak{D}) and Lemma 5.1 that

$$|G_1(t, s)| = |Z_1(t, \tau)\Phi_1(\tau, s)| \leq Ke^{-\alpha(t-\tau)}\rho(A) \leq Ke^{-\alpha(t-s)}e^{\alpha\theta}\rho(A).$$

If $s \in [t_r, \zeta_r]$ ($r = i + 1, \dots, j$), then $s - t_r \leq \theta$. In view of (\mathfrak{D}) and Lemma 5.1, we have

$$|G_1(t, s)| = |Z_1(t, t_r)\Phi_1(t_r, s)| \leq Ke^{-\alpha(t-t_r)}\rho(A) \leq Ke^{-\alpha(t-s)}e^{\alpha\theta}\rho(A).$$

If $s \in [\zeta_r, t_{r+1}]$ ($r = i, \dots, j - 1$), similar to the above inequality, we have the same conclusion.

If $s \in [\zeta_j, t]$, owing to **(A4)**, we have $t - s \leq \theta$. It follows from Lemma 5.1 and $K \geq 1$ that

$$|G_1(t, s)| = |\Phi_1(t, s)| \leq \rho(A) \leq Ke^{-\alpha(t-s)}e^{\alpha\theta}\rho(A). \quad (5.5)$$

Case 2. $t \leq \tau$.

By the definition of $G_1(t, s)$ we have $s \in [\min(t, \zeta_j), \max(\tau, \zeta_i)]$.

If $t \leq \zeta_j$, then $t < s$ which contradicts to our assumption that $t \geq s$. Thus, we just consider the case that $\zeta_j \leq t$. We divide the discussion into two subcases.

Subcase 2.1. $\zeta_j \leq t \leq t_{j+1} \leq \tau$.

For $t \geq s$, the only possibility is that $s \in [\zeta_j, t]$. Similar to (5.5), we have

$$|G_1(t, s)| = |\Phi_1(t, s)| \leq K e^{-\alpha(t-s)} e^{\alpha\theta} \rho(A).$$

Subcase 2.2. $\zeta_j \leq t \leq \tau \leq t_{j+1}$.

If $t \geq s$, then $s \in [\zeta_j, t]$ or $s \in [\zeta_j, \tau]$.

When $s \in [\zeta_j, t]$, similar to (5.5), we get

$$|G_1(t, s)| \leq K e^{-\alpha(t-s)} e^{\alpha\theta} \rho(A).$$

When $s \in [\zeta_j, \tau]$, we have $s \in \bar{I}_j$. Since $t \geq s$, following (D) and Lemma 5.1, we obtain

$$|G_1(t, s)| = |Z_1(t, \tau) \Phi_1(\tau, s)| = |Z_1(t, s) Z_1(s, \tau) \Phi_1(\tau, s)| \leq K e^{-\alpha(t-s)} \rho_0(A) \rho(A).$$

Note that $\tilde{\rho}(A) = \max(\rho(A) \rho_0(A), \rho(A) e^{\alpha\theta})$, we complete the proof. \square

Similar to Lemma 2.3, we have the following:

Lemma 5.3 *Assume that condition (D) holds, then*

$$\lim_{t \rightarrow -\infty} |Z_1(t, \tau)| = +\infty, \quad \lim_{t \rightarrow +\infty} |Z_2(t, \tau)| = +\infty, \quad \forall \tau \in \mathbb{R}.$$

Moreover, the unique bounded solution in \mathbb{R} of subsystem (2.9) (subsystem (2.10)) is trivial.

Proof The proof is similar to that of Lemma 2.3 and so it is omitted. \square

Lemma 5.4 ([33], Lemma 5.1) *Let $t \mapsto z(t, \tau, \xi)$ and $t \mapsto z(t, \tau, \xi')$ be the solutions of system (1.3) passing respectively through ξ and ξ' at $t = \tau$. If (3.5) is valid, then it follows that*

$$|z(t, \tau, \xi') - z(t, \tau, \xi)| \leq |\xi - \xi'| e^{p(\ell)|t-\tau|}$$

where $z(t, \cdot) = (x(t, \cdot), y(t, \cdot))^T$ and $p(\ell)$ is defined by

$$p(\ell) = \eta_1 + \frac{\eta_2 e^{\eta_1 \theta}}{1 - v} \quad \text{with} \quad \eta_1 = M + \ell, \quad \eta_2 = M_0 + \ell,$$

and $v \in [0, 1)$ is defined by (3.5).

Remark 5.3 If $h(t, z(t), z(\gamma(t))) = 0$, take $\ell = 0$, Lemma 5.4 reduces to Lemma 5.2 in [33]. Moreover, since $p(\ell) > p(0)$ and $F(\ell, \theta) \geq F(0, \theta)$ in (3.5), Lemma 5.4 is also valid for system (1.4).

Lemma 5.5 (DEPCAG Gronwall inequality [30, 31, 15]) *Let $\varrho, \eta : \mathbb{R} \rightarrow [0, \infty)$ be two functions such that u is continuous and η is locally integrable satisfying*

$$\bar{\theta} = \sup_{i \in \mathbb{Z}} \left\{ \theta_i : \theta_i := 2 \int_{I_i} \eta(s) ds \right\} < 1$$

Suppose that for $\tau \leq t$ or $t \leq \tau$, we have the inequality

$$\varrho(t) \leq \varrho(\tau) + \left| \int_{\tau}^t \eta(s) [\varrho(s) + \varrho(\gamma(s))] ds \right|.$$

Then

$$\begin{aligned} \varrho(t) &\leq \varrho(\tau) \exp \left\{ \tilde{\theta} \int_{\tau}^t \eta(s) ds \right\}, \\ \varrho(\gamma(t)) &\leq (1 - \bar{\theta})^{-1} \varrho(\tau) \exp \left\{ \tilde{\theta} \int_{\tau}^t \eta(s) ds \right\}, \end{aligned}$$

where $\tilde{\theta} = \frac{2 - \bar{\theta}}{1 - \bar{\theta}}$.

6 System (1.5) is topologically conjugate to system (1.4)

Suppose that $\begin{pmatrix} X(t, t_0, x_0) \\ Y(t, t_0, x_0, y_0) \end{pmatrix}$ is the solution of system (1.5) satisfying that $\begin{pmatrix} X(t_0) \\ Y(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and $\begin{pmatrix} u(t, t_0, \xi) \\ v(t, t_0, \eta) \end{pmatrix}$ is the solution of system (1.4) satisfying that $\begin{pmatrix} u(t_0) \\ v(t_0) \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, where $t_0 \in \mathbb{R}$, $x_0, \xi \in \mathbb{R}^{n_1}$, $y_0, \eta \in \mathbb{R}^{n_2}$.

Lemma 6.1 *For any $t \geq t_0$, the following inequalities hold:*

$$|X(t, t_0, x_0)| \leq |x_0| e^{-\alpha_0(t-t_0)},$$

$$|X(\gamma(t), t_0, x_0)| \leq (1 - \bar{\theta}) e^{\alpha_0 \bar{\theta}} |x_0| e^{-\alpha_0(t-t_0)},$$

where α_0 is defined in (3.4).

Proof From (2.7) we get

$$X(t, t_0, x_0) = Z_1(t, t_0) x_0 + \int_{t_0}^t G_1(t, s) f(s, X(s, t_0, x_0), X(\gamma(s), t_0, x_0)) ds.$$

It follows from condition (\mathfrak{D}) and Lemma 5.2 that

$$|X(t, t_0, x_0)| \leq e^{-\alpha(t-t_0)}|x_0| + l\tilde{\rho}(A) \int_{t_0}^t e^{-\alpha(t-s)}(|X(s)| + |X(\gamma(s))|)ds.$$

Thus

$$\begin{aligned} & e^{\alpha t}|X(t, t_0, x_0)| \\ & \leq e^{\alpha t_0}|x_0| + l\tilde{\rho}(A) \int_{t_0}^t (e^{\alpha s}|X(s)| + e^{\alpha\theta}e^{\alpha\gamma(s)}|X(\gamma(s))|)ds \\ & \leq e^{\alpha t_0}|x_0| + l\tilde{\rho}(A)e^{\alpha\theta} \int_{t_0}^t (e^{\alpha s}|X(s)| + e^{\alpha\gamma(s)}|X(\gamma(s))|)ds. \end{aligned}$$

Applying Lemma 5.5 to $\varrho(t) = e^{\alpha t}|X(t, t_0, x_0)|$ and $\eta(t) = 1$, we obtain that

$$|X(t, t_0, x_0)| \leq |x_0|e^{-\alpha(t-t_0)+2l\tilde{\rho}(A)e^{\alpha\theta}(t-t_0)}$$

and

$$|X(\gamma(t), t_0, x_0)| \leq (1 - \bar{\theta})|x_0|e^{-\alpha(\gamma(t)-t_0)+2l\tilde{\rho}(A)e^{\alpha\theta}(\gamma(t)-t_0)}.$$

Thus

$$|X(t, t_0, x_0)| \leq e^{-\alpha_0(t-t_0)}|x_0|,$$

and

$$|X(\gamma(t), t_0, x_0)| \leq (1 - \bar{\theta})e^{-\alpha_0(\gamma(t)-t_0)}|x_0| \leq (1 - \bar{\theta})e^{\alpha_0\theta}e^{-\alpha_0(t-t_0)}|x_0|. \square$$

Lemma 6.2 *For any fixed $t_0 \in \mathbb{R}$, $x_0, \xi \in \mathbb{R}^{n_1}$, there exists a unique $T(t_0, x_0)$ and $S(t_0, \xi) \in \mathbb{R}$, such that*

$$|X(T(t_0, x_0), t_0, x_0)| = 1, \quad T(t_0, x_0) \rightarrow -\infty \quad \text{when} \quad x_0 \rightarrow 0,$$

$$|u(S(t_0, \xi), t_0, \xi)| = 1, \quad S(t_0, \xi) \rightarrow -\infty \quad \text{when} \quad \xi \rightarrow 0.$$

Proof From Lemma 6.1, we have that $|X(t, t_0, x_0)| \leq |x_0|e^{-\alpha_0(t-t_0)}$ when $t \geq t_0$, where α_0 is defined in Lemma 6.1. If $x_0 \neq 0$ and $t \rightarrow +\infty$, then

$$|X(t, t_0, x_0)| \rightarrow 0.$$

If $t \geq \tau$,

$$|X(t, t_0, x_0)| = |X(t, \tau, X(\tau, t_0, x_0))| \leq |X(\tau, t_0, x_0)|e^{-\alpha_0(t-\tau)}. \quad (6.1)$$

Thus, for the fixed t_0 and x_0 , $|X(t, t_0, x_0)|$ is a strictly monotonous decreasing function about t . If t is fixed and $\tau \rightarrow -\infty$, then

$$e^{-\alpha_0(t-\tau)} \rightarrow 0.$$

Thus

$$|X(\tau, t_0, x_0)| \rightarrow +\infty \quad \text{when} \quad \tau \rightarrow -\infty.$$

Therefore, there exists a unique time $T(t_0, x_0)$ such that $|X(T(t_0, x_0), t_0, x_0)| = 1$. Moreover, when $x_0 \rightarrow 0$, $T(t_0, x_0) \rightarrow -\infty$.

By condition \mathfrak{D} , for $t > t_0$, we have

$$|u(t, t_0, \xi)| = |Z(t, t_0)\xi| \leq e^{-\alpha(t-t_0)}|\xi|.$$

Thus when $t \rightarrow +\infty$,

$$|Z(t, t_0)\xi| \rightarrow 0.$$

Similar to (6.1), we could obtain that for fixed t_0 and ξ , $|Z(t, t_0)\xi|$ is a strictly monotonous decreasing function about t . Moreover, when $t \rightarrow -\infty$,

$$|Z(t, t_0)\xi| \rightarrow +\infty.$$

Therefore, for a fixed $\xi \in \mathbb{R}^{n_1}$, $\xi \neq 0$, there exists a unique time $S(t_0, \xi)$ such that

$$|Z(S(t_0, \xi), t_0, \xi)| = 1,$$

and

$$S(t_0, \xi) \rightarrow -\infty \quad \text{when} \quad \xi \rightarrow 0. \quad \square$$

Lemma 6.3 *For any $x_0 \neq 0$, $\xi \neq 0$ and $t \in \mathbb{R}$, we have*

$$T(t, X(t, t_0, x_0)) = T(t_0, x_0),$$

$$S(t, u(t, \tau, \xi)) = S(\tau, \xi).$$

Proof It follows from Lemma 6.2 that

$$1 = |X(T(t, X(t, t_0, x_0)), t, X(t, t_0, x_0))| = |X(T(t, X(t, t_0, x_0)), t_0, x_0)|.$$

From $|X(T(t_0, x_0), t_0, x_0)| = 1$ and Lemma 6.2, we get

$$T(t, X(t, t_0, x_0)) = T(t_0, x_0).$$

The second equality can be proved in a similar way. \square

Lemma 6.4 *For any $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^{n_1}$, the following inequality holds.*

$$\left| \int_{t_0}^{+\infty} G_2(t_0, s)g(s, X(s, t_0, x_0), X(\gamma(s), t_0, x_0))ds \right| \leq K\lambda\tilde{\rho}(B)((\alpha + \rho_0)^{-1} + \alpha^{-1})|x_0|.$$

Proof From Lemmas 5.2 and 6.1, we get

$$\begin{aligned}
& \left| \int_{t_0}^{+\infty} G_2(t_0, s) g(s, X(s, t_0, x_0), X(\gamma(s), t_0, x_0)) ds \right| \\
& \leq \int_{t_0}^{+\infty} K \lambda \tilde{\rho}(B) e^{\alpha(t_0-s)} (|X(s, t_0, x_0)| + |X(\gamma(s), t_0, x_0)|) ds \\
& \leq \int_{t_0}^{+\infty} K \lambda \tilde{\rho}(B) e^{\alpha(t_0-s)} e^{-\alpha_0(s-t_0)} (|x_0| + (1 - \bar{\theta}) e^{\alpha_0 \theta} |x_0|) ds \\
& \leq K \lambda (\alpha + \alpha_0)^{-1} \tilde{\rho}(B) (1 + (1 - \bar{\theta}) e^{\alpha_0 \theta}) |x_0|.
\end{aligned}$$

Definition 6.2 For any $t \in \mathbb{R}$, $\xi \in \mathbb{R}^{n_1}$ and $\eta \in \mathbb{R}^{n_2}$, we define $L_1 : \mathbb{R} \times \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_1}$, $L_2 : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_2}$ and $L : \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^n$ as follows:

$$L_1(t, \xi) = \begin{cases} X(t, S(t, \xi), u(S(t, \xi), t, \xi)) & \xi \neq 0, \\ 0 & \xi = 0, \end{cases}$$

$$L_2(t, \xi, \eta) = \eta - \int_t^{+\infty} G_2(t, s) g\left(s, X(s, t, L_1(t, \xi)), X(\gamma(s), t, L_1(t, \xi))\right) ds,$$

and

$$L(t, \xi, \eta) = \begin{pmatrix} L_1(t, \xi) \\ L_2(t, \xi, \eta) \end{pmatrix}.$$

Lemma 6.5 $L_1(t, \xi)$ is a continuous function of ξ and $L_1(t, u(t, \tau, \xi)) = X(t, \tau, L_1(\tau, \xi))$.

Proof By Lemma 6.2, we have

$$S(t, \xi) \rightarrow -\infty \quad \text{when} \quad \xi \rightarrow 0.$$

When $\xi \rightarrow 0$, it follows from Lemma 6.1 that

$$|X(t, S(t, \xi), u(S(t, \xi), t, \xi))| \leq |u(S(t, \xi), t, \xi)| e^{-\alpha_0(t-S(t, \xi))} = e^{-\alpha_0(t-S(t, \xi))} \rightarrow 0.$$

Hence, $L_1(t, \xi)$ is a continuous function of ξ .

Furthermore, from Lemma 6.3, we have that

$$\begin{aligned}
L_1(t, u(t, \tau, \xi)) &= X(t, S(t, u(t, \tau, \xi)), u(S(t, u(t, \tau, \xi)), t, u(t, \tau, \xi))) \\
&= X(t, S(\tau, \xi), u(S(\tau, \xi), \tau, \xi)) \\
&= X(t, \tau, X(\tau, S(\tau, \xi), u(S(\tau, \xi), \tau, \xi))) \\
&= X(t, \tau, L_1(\tau, \xi)). \quad \square
\end{aligned}$$

Lemma 6.6 $\begin{pmatrix} L_1(t, u(t, \tau, \xi)) \\ L_2(t, u(t, \tau, \xi), v(t, \tau, \eta)) \end{pmatrix} = \begin{pmatrix} X(t, \tau, L_1(\tau, \xi)) \\ Y(t, \tau, L_1(\tau, \xi), L_2(\tau, \xi, \eta)) \end{pmatrix}.$

Proof Due to Lemma 6.5, we get

$$L_1(t, u(t, \tau, \xi)) = X(t, \tau, L_1(\tau, \xi)).$$

$$\begin{aligned} & L_2(t, u(t, \tau, \xi), v(t, \tau, \eta)) \\ &= v(t, \tau, \eta) - \int_t^{+\infty} G_2(t, s) g\left(s, X(s, t, L_1(t, u(t, \tau, \xi))), X(\gamma(s), t, L_1(t, u(t, \tau, \xi)))\right) ds \\ &= v(t, \tau, \eta) - \int_t^{+\infty} G_2(t, s) g\left(s, X(s, t, X(t, \tau, L_1(\tau, \xi))), X(\gamma(s), t, X(t, \tau, L_1(\tau, \xi)))\right) ds \\ &= v(t, \tau, \eta) - \int_t^{+\infty} G_2(t, s) g\left(s, X(s, \tau, L_1(\tau, \xi)), X(\gamma(s), \tau, L_1(\tau, \xi))\right) ds. \end{aligned} \quad (6.2)$$

Denote $J(t) = - \int_t^{+\infty} G_2(t, s) g\left(s, X(s, \tau, L_1(\tau, \xi)), X(\gamma(s), \tau, L_1(\tau, \xi))\right) ds$. Suppose $t \in I_j$, we obtain

$$\begin{aligned} J'(t) &= -B(t) \int_t^{+\infty} G_2(t, s) g\left(s, X(s, \tau, L_1(\tau, \xi)), X(\gamma(s), \tau, L_1(\tau, \xi))\right) ds \\ &\quad - B_0(t) \int_{\gamma(t)}^{+\infty} G_2(t, s) g\left(s, X(s, \tau, L_1(\tau, \xi)), X(\gamma(s), \tau, L_1(\tau, \xi))\right) ds \\ &\quad + g(t, X(t, \tau, L_1(\tau, \xi)), X(\gamma(t), \tau, L_1(\tau, \xi))). \end{aligned}$$

Furthermore, from (6.2), we have

$$\begin{aligned} & L_2'(t, u(t, \tau, \xi), v(t, \tau, \eta)) \\ &= B(t) L_2(t, u(t, \tau, \xi), v(t, \tau, \eta)) + B_0(t) L_2(\gamma(t), u(t, \tau, \xi), v(t, \tau, \eta)) \\ &\quad + g(t, X(t, \tau, L_1(\tau, \xi)), X(\gamma(t), \tau, L_1(\tau, \xi))). \end{aligned}$$

Thus $\begin{pmatrix} L_1(t, u(t, \tau, \xi), v(t, \tau, \eta)) \\ L_2(t, u(t, \tau, \xi), v(t, \tau, \eta)) \end{pmatrix}$ is a solution of system (1.5).

From

$$\begin{pmatrix} L_1(t, u(t, \tau, \xi), v(t, \tau, \eta)) \\ L_2(t, u(t, \tau, \xi), v(t, \tau, \eta)) \end{pmatrix} \Big|_{t=\tau} = \begin{pmatrix} L_1(\tau, \xi) \\ L_2(\tau, \xi, \eta) \end{pmatrix}$$

and

$$\begin{pmatrix} X(t, \tau, L_1(\tau, \xi)) \\ Y(t, \tau, L_1(\tau, \xi), L_2(\tau, \xi, \eta)) \end{pmatrix} \Big|_{t=\tau} = \begin{pmatrix} L_1(\tau, \xi) \\ L_2(\tau, \xi, \eta) \end{pmatrix},$$

we get the conclusion of the lemma. \square

Definition 6.3 For any $t \in \mathbb{R}$, $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$, we denote $H(t, x, y) = \begin{pmatrix} H_1(t, x) \\ H_2(t, x, y) \end{pmatrix}$, where $H_1(t, x)$ and $H_2(t, x, y)$ are defined as

$$H_1(t, x) = \begin{cases} u(t, T(t, x), X(T(t, x), t, x)) & x \neq 0, \\ 0 & x = 0, \end{cases}$$

and

$$H_2(t, x, y) = y + \int_t^{+\infty} G_2(t, s)g(s, X(s, t, x), X(\gamma(s), t, x))ds.$$

Lemma 6.7 $H_1(t, x)$ is a continuous function of x .

Proof From (5.1), we get

$$u(t, T(t, x), X(T(t, x), t, x)) = Z_1(t, T(t, x))X(T(t, x), t, x),$$

which together with condition (\mathfrak{D}) implies that

$$|u(t, T(t, x), X(T(t, x), t, x))| \leq e^{-\alpha(t-T(t, x))}|X(T(t, x), t, x)| \leq e^{-\alpha(t-T(t, x))}, \quad t \geq T(t, x).$$

From Lemma 6.2, we have that

$$T(t, x) \rightarrow -\infty, \quad \text{when } x \rightarrow 0.$$

Thus $H_1(t, x)$ is a continuous function of x . \square

Lemma 6.8

$$\begin{pmatrix} H_1(t, X(t, t_0, x_0)) \\ H_2(t, X(t, t_0, x_0), Y(t, t_0, x_0, y_0)) \end{pmatrix} = \begin{pmatrix} u(t, t_0, H_1(t_0, x_0)) \\ v(t, t_0, H_2(t, x_0, y_0)) \end{pmatrix}.$$

Proof From Lemma 6.3, we have

$$\begin{aligned} H_1(t, X(t, t_0, x_0)) &= u(t, T(t, X(t, t_0, x_0)), X(T(t, X(t, t_0, x_0)), t, X(t, t_0, x_0))) \\ &= u(t, T(t_0, x_0), X(T(t_0, x_0), t_0, x_0)) \\ &= u(t, t_0, u(t_0, T(t_0, x_0), X(T(t_0, x_0), t_0, x_0))) \\ &= u(t, t_0, H_1(t_0, x_0)). \end{aligned}$$

$$\begin{aligned} &H_2(t, X(t, t_0, x_0), Y(t, t_0, x_0, y_0)) \\ &= Y(t, t_0, x_0, y_0) + \int_t^{+\infty} G_2(t, s)g\left(s, X(s, t, X(t, t_0, x_0)), X(\gamma(s), t, X(t, t_0, x_0))\right)ds \\ &= Y(t, t_0, x_0, y_0) + \int_t^{+\infty} G_2(t, s)g(s, X(s, t_0, x_0), X(\gamma(s), t_0, x_0))ds \end{aligned}$$

Since

$$\begin{aligned} & H_2'(t, X(t, t_0, x_0)) \\ &= B(t)H_2(t, X(t, t_0, x_0), Y(t, t_0, x_0, y_0)) + B_0(t)H_2(\gamma(t), X(\gamma(t), t_0, x_0), Y(\gamma(t), t_0, x_0, y_0)), \end{aligned}$$

$H_2(t, X(t, t_0, x_0), Y(t, t_0, x_0, y_0))$ is a solution of system (2.10).

Moreover,

$$H_2(t, X(t, t_0, x_0), Y(t, t_0, x_0, y_0))|_{t=t_0} = H_2(t_0, x_0, y_0).$$

Thus $H_2(t, X(t, t_0, x_0), Y(t, t_0, x_0, y_0)) = v(t, t_0, H_2(t_0, x_0, y_0))$. \square

Lemma 6.9 For any $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^{n_1}$, $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^{n_1}$, we have

$$S(t_0, H_1(t_0, x_0)) = T(t_0, x_0), \quad T(\tau, L_1(\tau, \xi)) = S(\tau, \xi).$$

Proof From the definition of H_1 , we have

$$\begin{aligned} 1 &= |u(S(t_0, H_1(t_0, x_0)), t_0, H_1(t_0, x_0))| \\ &= |u(S(t_0, H_1(t_0, x_0)), t_0, u(t_0, T(t_0, x_0), X(T(t_0, x_0), t_0, x_0)))| \\ &= |u(S(t_0, H_1(t_0, x_0)), T(t_0, x_0), X(T(t_0, x_0), t_0, x_0))|, \end{aligned}$$

which implies that

$$S(t_0, H_1(t_0, x_0)) = S(T(t_0, x_0), X(T(t_0, x_0), t_0, x_0)).$$

From

$$|u(T(t_0, x_0), T(t_0, x_0), X(T(t_0, x_0), t_0, x_0))| = |X(T(t_0, x_0), t_0, x_0)| = 1,$$

we obtain that

$$S(T(t_0, x_0), X(T(t_0, x_0), t_0, x_0)) = T(t_0, x_0).$$

Thus

$$S(t_0, H_1(t_0, x_0)) = T(t_0, x_0).$$

Similarly, we could prove that $T(\tau, L_1(\tau, \xi)) = S(\tau, \xi)$. \square

Lemma 6.10 For any $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^{n_1}$ and $y_0 \in \mathbb{R}^{n_2}$, we have

$$L(t_0, H(t_0, x_0)) = (x_0, y_0)^T.$$

Proof If $x_0 = 0$, it is easy to see that $L_1(t_0, H_1(t_0, x_0)) = x_0$.

If $x_0 \neq 0$, from Lemma 6.9 and the definitions of L_1 and H_1 , we get

$$\begin{aligned}
L_1(t_0, H_1(t_0, x_0)) &= X\left(t_0, S(t_0, H_1(t_0, x_0)), u(S(t_0, H_1(t_0, x_0)), t_0, H_1(t_0, x_0))\right) \\
&= X\left(t_0, T(t_0, x_0), u(T(t_0, x_0), t_0, u(t_0, T(t_0, x_0), X(T(t_0, x_0), t_0, x_0)))\right) \\
&= X\left(t_0, T(t_0, x_0), u(T(t_0, x_0), T(t_0, x_0), X(T(t_0, x_0), t_0, x_0))\right) \\
&= X(t_0, T(t_0, x_0), X(T(t_0, x_0), t_0, x_0)) \\
&= x_0,
\end{aligned}$$

which together with the definitions of L_2 and H_2 implies that

$$\begin{aligned}
&L_2(t_0, H_1(t_0, x_0), H_2(t_0, x_0, y_0)) \\
&= H_2(t_0, x_0, y_0) - \int_{t_0}^{+\infty} G_2(t_0, s)g\left(s, X(s, t_0, L_1(t_0, H_1(t_0, x_0))), X(\gamma(s), t_0, L_1(t_0, H_1(t_0, x_0)))\right)ds \\
&= y_0 + \int_{t_0}^{+\infty} G_2(t_0, s)g(s, X(s, t_0, x_0), X(\gamma(s), t_0, x_0))ds \\
&\quad - \int_{t_0}^{+\infty} G_2(t_0, s)g(s, X(s, t_0, x_0), X(\gamma(s), t_0, x_0))ds \\
&= y_0. \quad \square
\end{aligned}$$

Lemma 6.11 For any $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^{n_1}$ and $\eta \in \mathbb{R}^{n_2}$, we have

$$H(\tau, L(\tau, \xi, \eta)) = (\xi, \eta)^T.$$

Proof If $\xi = 0$, it is obvious that $H_1(\tau, L_1(\tau, \xi)) = \xi$.

If $\xi \neq 0$, by Lemma 6.9 and the definitions of H_1 and L_1 , we obtain

$$\begin{aligned}
H_1(\tau, L_1(\tau, \xi)) &= u\left(\tau, T(\tau, L_1(\tau, \xi)), X(T(\tau, L_1(\tau, \xi)), \tau, L_1(\tau, \xi))\right) \\
&= u\left(\tau, S(\tau, \xi), X(S(\tau, \xi), \tau, L_1(\tau, \xi))\right) \\
&= u\left(\tau, S(\tau, \xi), X(S(\tau, \xi), \tau, X(\tau, S(\tau, \xi), u(S(\tau, \xi), \tau, \xi)))\right) \\
&= u(\tau, S(\tau, \xi), u(S(\tau, \xi), \tau, \xi)) \\
&= \xi.
\end{aligned}$$

In what follows, we prove that $H_2(\tau, L_1(\tau, \xi), L_2(\tau, \xi, \eta)) = \eta$.

For any $t \in \mathbb{R}$, $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$, due to Lemma 6.4, we have

$$\begin{aligned}
|H_2(t, x, y) - y| &= \left| \int_t^{+\infty} G_2(t, s)g(s, X(s, t, x), X(\gamma(s), t, x))ds \right| \\
&\leq K\lambda\tilde{\rho}(B)((\alpha + \rho_0)^{-1} + \alpha^{-1})|x_0|.
\end{aligned}$$

From Lemma 6.4 and the definition of L_2 , we obtain

$$\begin{aligned} |L_2(t, \xi, \eta) - \eta| &\leq \left| \int_t^{+\infty} G_2(t, s) g\left(s, X(s, t, L_1(t, \xi)), X(\gamma(s), t, L_1(t, \xi))\right) ds \right| \\ &\leq K\lambda\tilde{\rho}(B)((\alpha + \rho_0)^{-1} + \alpha^{-1})|L_1(t, \xi)|. \end{aligned}$$

Thus, by Lemma 6.6 we get

$$\begin{aligned} J &\triangleq |H_2(t, L_1(t, u(t, \tau, \xi)), L_2(t, u(t, \tau, \xi), v(t, \tau, \eta))) - v(t, \tau, \eta)| \\ &\leq |H_2(t, L_1(t, u(t, \tau, \xi)), L_2(t, u(t, \tau, \xi), v(t, \tau, \eta))) - L_2(t, u(t, \tau, \xi), v(t, \tau, \eta))| \\ &\quad + |L_2(t, u(t, \tau, \xi), v(t, \tau, \eta)) - v(t, \tau, \eta)| \\ &\leq 2K\lambda\tilde{\rho}(B)((\alpha + \rho_0)^{-1} + \alpha^{-1})|L_1(t, u(t, \tau, \xi))| \\ &\leq 2K\lambda\tilde{\rho}(B)((\alpha + \rho_0)^{-1} + \alpha^{-1})|X(t, \tau, L_1(\tau, \xi))|. \end{aligned}$$

It follows from Lemma 6.1 that

$$J \leq 2K\lambda\tilde{\rho}(B)((\alpha + \rho_0)^{-1} + \alpha^{-1})|L_1(\tau, \xi)|e^{-\alpha_0(t-\tau)}, \quad t \geq \tau. \quad (6.3)$$

From (5.2), Lemmas 6.6 and 6.8, we have

$$\begin{aligned} &H_2(t, L_1(t, u(t, \tau, \xi)), L_2(t, u(t, \tau, \xi), v(t, \tau, \eta))) \\ &= H_2(t, X(t, \tau, L_1(\tau, \xi)), Y(t, \tau, L_1(\tau, \xi)), L_2(\tau, \xi, \eta)) \\ &= v(t, \tau, H_2(t, L_1(\tau, \xi), L_2(\tau, \xi, \eta))) \\ &= Z_2(t, \tau)H_2(\tau, L_1(\tau, \xi), L_2(\tau, \xi, \eta)). \end{aligned}$$

By (6.3) and $v(t, \tau, \eta) = Z_2(t, \tau)\eta$, we get

$$\begin{aligned} &|Z_2(t, \tau) \cdot (H_2(\tau, L_1(\tau, \xi), L_2(\tau, \xi, \eta)) - \eta)| \\ &= |H_2(t, L_1(t, u(t, \tau, \xi)), L_2(t, u(t, \tau, \xi), v(t, \tau, \eta))) - v(t, \tau, \eta)| \\ &\leq 2K\lambda\tilde{\rho}(B)((\alpha + \rho_0)^{-1} + \alpha^{-1})|L_1(\tau, \xi)|e^{-\alpha_0(t-\tau)}, \quad t \geq \tau. \end{aligned}$$

For fixed τ and ξ , $L_1(\tau, \xi)$ is a fixed value. Thus the above equality is bounded when $t \geq \tau$. Moreover, it follows from condition (\mathfrak{D}) that the above equality is bounded when $t \leq \tau$. Therefore, $Z_2(t, \tau) \cdot (H_2(\tau, L_1(\tau, \xi), L_2(\tau, \xi, \eta)) - \eta)$ is a bounded solution of system (2.10).

Since system (2.10) has an α -exponential dichotomy, for fixed τ , ξ and η , it has a unique bounded solution, zero solution. Thus

$$H_2(\tau, L_1(\tau, \xi), L_2(\tau, \xi, \eta)) - \eta = 0.$$

That is $H_2(\tau, L_2(\tau, \xi, \eta)) = \eta$. \square

Lemma 6.12 *System (1.5) is topologically conjugate to system (1.4).*

Proof It follows from Lemmas 6.10 and 6.11 that for a fixed t , $H(t, x, y) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^n$ is a bijection and $H^{-1}(t, x, y) = L(t, x, y)$.

According to Lemma 5.4 and Remark 5.3, solutions of systems (1.5) and (1.4) are continuous with respect to initial values.

By the definitions of $H(t, \cdot)$ and $L(t, \cdot)$, and lemmas 6.5 and 6.7, we get that both $H(t, \cdot)$ and $L(t, \cdot)$ are continuous. Thus $H(t, \cdot)$ and $L(t, \cdot)$ are homeomorphisms of \mathbb{R}^n .

Moreover, Lemmas 6.6 and 6.8 imply that $H(t, \cdot)$ sends the solutions of system (1.5) onto those of system (1.4) and $L(t, \cdot)$ sends the solutions of system (1.4) onto those of system (1.5). Therefore, system (1.5) and system (1.4) are topologically conjugated. \square

7 System (1.3) is topologically conjugate to system (1.5)

First we introduce a new system

$$\begin{cases} x' = A(t)x(t) + A_0(t)x(\gamma(t)) + f(t, x(t), x(\gamma(t))) + p(t, y(t), y(\gamma(t))) \\ y' = B(t)y(t) + B_0(t)y(\gamma(t)) + g(t, x(t), x(\gamma(t))) + q(t, y(t), y(\gamma(t))), \end{cases} \quad (7.1)$$

where $f(t, \cdot)$ and $g(t, \cdot)$ are defined in system (1.3), $p : \mathbb{R} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_1}$ and $q : \mathbb{R} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_2}$ satisfying that for the δ and ω in (\mathfrak{B}_2) and any $t \in \mathbb{R}$, $y_1, y_2, \bar{y}_1, \bar{y}_2 \in \mathbb{R}^{n_2}$ such that

$$\begin{aligned} |p(t, y_1, y_2)| &\leq \delta, \quad |q(t, y_1, y_2)| \leq \delta, \\ |p(t, y_1, y_2) - p(t, \bar{y}_1, \bar{y}_2)| &\leq \omega(|y_1 - \bar{y}_1| + |y_2 - \bar{y}_2|), \end{aligned}$$

$$|q(t, y_1, y_2) - q(t, \bar{y}_1, \bar{y}_2)| \leq \omega(|y_1 - \bar{y}_1| + |y_2 - \bar{y}_2|).$$

Lemma 7.1 *If (3.3) holds, then there exists a unique function $\bar{H}(t, x, y) : \mathbb{R} \times \mathbb{R}^{n_1+n_2} \rightarrow \mathbb{R}^{n_1+n_2}$ satisfying that*

(i) *There exists a constant $\bar{\sigma} > 0$ such that*

$$|\bar{H}(t, x, y) - (x, y)^T| \leq \bar{\sigma}.$$

(ii) *If $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a solution of system (1.3), then $\bar{H}(t, x(t), y(t))$ is a solution of system (7.1).*

Proof For any fixed $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^{n_1}$ and $\eta \in \mathbb{R}^{n_2}$, suppose that $\begin{pmatrix} x(t, \tau, \xi, \eta) \\ y(t, \tau, \xi, \eta) \end{pmatrix}$ is a solution of system (1.3) satisfying $\begin{pmatrix} x(\tau, \tau, \xi, \eta) \\ y(\tau, \tau, \xi, \eta) \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$.

Denote $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$ where $z_1(t) \in \mathbb{R}^{n_1}$ and $z_2(t) \in \mathbb{R}^{n_2}$, $W(t) = \begin{bmatrix} A(t) & \\ & B(t) \end{bmatrix}$, $W_0(t) = \begin{bmatrix} A_0(t) & \\ & B_0(t) \end{bmatrix}$ and

$$\begin{aligned} & \bar{h}(t, z(t), z(\gamma(t)), (\tau, \xi, \eta)) \\ &= \begin{pmatrix} \bar{h}_1(t, z(t), z(\gamma(t)), (\tau, \xi, \eta)) \\ \bar{h}_2(t, z(t), z(\gamma(t)), (\tau, \xi, \eta)) \end{pmatrix} \\ &= \begin{pmatrix} f(t, x(t, \tau, \xi, \eta) + z_1(t), x(\gamma(t), \tau, \xi, \eta) + z_1(\gamma(t))) \\ g(t, x(t, \tau, \xi, \eta) + z_1(t), x(\gamma(t), \tau, \xi, \eta) + z_1(\gamma(t))) \end{pmatrix} \\ &+ \begin{pmatrix} p(t, y(t, \tau, \xi, \eta) + z_2(t), y(\gamma(t), \tau, \xi, \eta) + z_2(\gamma(t))) \\ q(t, y(t, \tau, \xi, \eta) + z_2(t), y(\gamma(t), \tau, \xi, \eta) + z_2(\gamma(t))) \end{pmatrix} \\ &+ \begin{pmatrix} -f(t, x(t, \tau, \xi, \eta), x(\gamma(t), \tau, \xi, \eta)) - \phi(t, y(t, \tau, \xi, \eta), y(\gamma(t), \tau, \xi, \eta)) \\ -g(t, x(t, \tau, \xi, \eta), x(\gamma(t), \tau, \xi, \eta)) - \psi(t, y(t, \tau, \xi, \eta), y(\gamma(t), \tau, \xi, \eta)) \end{pmatrix}. \end{aligned}$$

From

$$|\bar{h}(t, z(t), z(\gamma(t)), (\tau, \xi, \eta))| \leq 2\lambda|z(t)| + 2\lambda|z(\gamma(t))| + 4\delta,$$

$$\begin{aligned} & |\bar{h}(t, z(t), z(\gamma(t)), (\tau, \xi, \eta)) - \bar{h}(t, \bar{z}(t), \bar{z}(\gamma(t)), (\tau, \xi, \eta))| \\ & \leq 2\omega|z(t) - \bar{z}(t)| + 2\omega|z(\gamma(t)) - \bar{z}(\gamma(t))|, \end{aligned}$$

and Theorem 1, we get that system

$$z'(t) = W(t)z(t) + W_0(t)z(\gamma(t)) + \bar{h}(t, z(t), z(\gamma(t)), (\tau, \xi, \eta)) \quad (7.2)$$

has a unique bounded solution for fixed τ , ξ and η . We denote by

$$\chi(t, (\tau, \xi, \eta)) = \begin{pmatrix} \chi_1(t, (\tau, \xi, \eta)) \\ \chi_2(t, (\tau, \xi, \eta)) \end{pmatrix} \quad \text{and} \quad |\chi(t, (\tau, \xi, \eta))| \leq \bar{\sigma},$$

where $\chi_1(t, (\tau, \xi, \eta)) \in \mathbb{R}^{n_1}$ and $\chi_2(t, (\tau, \xi, \eta)) \in \mathbb{R}^{n_2}$.

For any $t \in \mathbb{R}$, $\xi \in \mathbb{R}^{n_1}$ and $\eta \in \mathbb{R}^{n_2}$, define

$$\bar{H}(t, \xi, \eta) = \begin{pmatrix} \bar{H}_1(t, \xi, \eta) \\ \bar{H}_2(t, \xi, \eta) \end{pmatrix} = \begin{pmatrix} \xi + \chi_1(t, (t, \xi, \eta)) \\ \eta + \chi_2(t, (t, \xi, \eta)) \end{pmatrix}.$$

Thus $\bar{H}(t, \xi, \eta)$ is continuous on $\mathbb{R} \times \mathbb{R}^{n_1+n_2}$ and

$$\left| \bar{H}(t, \xi, \eta) - \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right| \leq \bar{\sigma}.$$

Moreover,

$$\bar{H}(t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta)) = \begin{pmatrix} x(t, \tau, \xi, \eta) + \chi_1(t, (t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))) \\ y(t, \tau, \xi, \eta) + \chi_2(t, (t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))) \end{pmatrix},$$

where $\chi(s, (t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta)))$ is the unique bounded solution of system

$$\frac{dz}{ds} = W(s)z(s) + W_0(s)z(\gamma(s)) + \bar{h}(s, z(s), z(\gamma(s)), (t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))).$$

From

$$\begin{aligned} x(s, (t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))) &= x(s, \tau, \xi, \eta), \\ y(s, (t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))) &= y(s, \tau, \xi, \eta), \end{aligned}$$

we have

$$\bar{h}(s, z(s), z(\gamma(s)), (t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))) = \bar{h}(s, z(s), z(\gamma(s)), (\tau, \xi, \eta)).$$

Thus

$$\chi(s, (t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))) = \chi(s, (\tau, \xi, \eta)), \quad \forall s \in \mathbb{R}.$$

Taking $s = t$, we get

$$\chi(t, (t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))) = \chi(t, (\tau, \xi, \eta)).$$

$$\text{Therefore, } \bar{H}(t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta)) = \begin{pmatrix} x(t, \tau, \xi, \eta) + \chi_1(t, (\tau, \xi, \eta)) \\ y(t, \tau, \xi, \eta) + \chi_2(t, (\tau, \xi, \eta)) \end{pmatrix}.$$

We could check that $\bar{H}(t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))$ is a solution of system (7.1) and $|\bar{H}(t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta)) - (x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))^T|$ is bounded. Therefore $\bar{H}(t, x, y)$ satisfies (i) and (ii).

Assume that $\bar{K}(t, x, y) = \begin{pmatrix} \bar{K}_1(t, x, y) \\ \bar{K}_2(t, x, y) \end{pmatrix}$ satisfies (i) and (ii), too, where $\bar{K}_1(t, x, y) \in \mathbb{R}^{n_1}$ and $\bar{K}_2(t, x, y) \in \mathbb{R}^{n_2}$. Since $\begin{pmatrix} x(t, \tau, \xi, \eta) \\ y(t, \tau, \xi, \eta) \end{pmatrix}$ is the solution of system (1.3), $\bar{K}(t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta))$ is a solution of system (7.1).

$$\text{Denote } w(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} \bar{K}_1(t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta)) - x(t, \tau, \xi, \eta) \\ \bar{K}_2(t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta)) - y(t, \tau, \xi, \eta) \end{pmatrix}.$$

From $w'(t) = W(t)w(t) + W_0(t)w(\gamma(t)) + \bar{h}(t, w(t), w(\gamma(t)), (\tau, \xi, \eta))$, we have that $w(t)$ is a bounded solution of system (7.2). Therefore

$$w(t) = \chi(t, (\tau, \xi, \eta)).$$

$$\text{Thus } \bar{K}(t, x(t, \tau, \xi, \eta), y(t, \tau, \xi, \eta)) = \begin{pmatrix} x(t, \tau, \xi, \eta) + \chi_1(t, (\tau, \xi, \eta)) \\ y(t, \tau, \xi, \eta) + \chi_2(t, (\tau, \xi, \eta)) \end{pmatrix}.$$

Taking $t = \tau$, we have

$$\bar{K}(\tau, \xi, \eta) = \begin{pmatrix} \xi + \chi_1(\tau, (\tau, \xi, \eta)) \\ \eta + \chi_2(\tau, (\tau, \xi, \eta)) \end{pmatrix} = \bar{H}(\tau, \xi, \eta).$$

Thus $\bar{H}(t, x, y)$ is a unique function satisfying the conditions (i) and (ii). We complete the proof. \square

Lemma 7.2 *System (1.5) is topologically conjugate to system (1.3).*

Proof From Lemma 7.1, for any $t \in \mathbb{R}$, $x, \tilde{x} \in \mathbb{R}^{n_1}$ and $y, \tilde{y} \in \mathbb{R}^{n_2}$, there exists a unique function $\tilde{H}(t, x, y)$ satisfies that

(i) There exists a const number $\sigma_1 > 0$ such that

$$|\tilde{H}(t, x, y) - (x, y)^T| \leq \sigma_1.$$

(ii) If $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a solution of system (1.3), then $H(t, x(t), y(t))$ is a solution of system (1.5).

Similarly, there exists a unique function $\tilde{L}(t, \tilde{x}, \tilde{y})$ satisfies that

(i) There exists a const number $\sigma_2 > 0$ such that

$$|\tilde{L}(t, \tilde{x}, \tilde{y}) - (\tilde{x}, \tilde{y})^T| \leq \sigma_2.$$

(ii) If $\begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{pmatrix}$ is a solution of system (1.5), then $\tilde{L}(t, \tilde{x}(t), \tilde{y}(t))$ is a solution of system (1.3).

In what followings, we prove that $\tilde{L}(t, \tilde{H}(t, x, y)) = (x, y)^T$ and $\tilde{H}(t, \tilde{L}(t, x, y)) = (x, y)^T$.

Denote $\tilde{J}(t, x, y) = \tilde{L}(t, \tilde{H}(t, x, y))$.

If $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a solution of system (1.3), then $\tilde{H}(t, x(t), y(t))$ is a solution of system (1.5). Thus $\tilde{L}(t, \tilde{H}(t, x(t), y(t)))$ is a solution of system (1.3). By a simple calculation, we get

$$|\tilde{J}(t, x, y) - (x, y)^T| \leq |\tilde{L}(t, \tilde{H}(t, x, y)) - \tilde{H}(t, x, y)| + |\tilde{H}(t, x, y) - (x, y)^T| \leq \sigma_1 + \sigma_2.$$

Therefore $\tilde{J}(t, x, y)$ is the unique function satisfying the conditions (i) and (ii) in Lemma 7.1 which transforms the solution of system (1.5) to those of itself.

In particular, taking $p = \phi$ and $q = \psi$ in system (7.1), then system (7.1) becomes system (1.3). From system (1.3) to itself, for any $t \in \mathbb{R}, x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}$, the function $\tilde{H}(t, x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfies the conditions (i) and (ii) in Lemma 7.1. Thus, for any $t \in \mathbb{R}, x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$,

$$\tilde{J}(t, x, y) = \tilde{H}(t, x, y) = \begin{pmatrix} x \\ y \end{pmatrix}.$$

That is

$$\tilde{L}(t, \tilde{H}(t, x, y)) = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}.$$

Applying Lemma 7.1 to system (7.1) with $p = 0$ and $q = 0$, we could prove that

$$\tilde{H}(t, \tilde{L}(t, \tilde{x}, \tilde{y})) = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \quad \forall t \in \mathbb{R}, \tilde{x} \in \mathbb{R}^{n_1}, \tilde{y} \in \mathbb{R}^{n_2}.$$

Therefore, for a fixed t , $\tilde{H}^{-1}(t, \cdot, \cdot) = \tilde{L}(t, \cdot, \cdot)$.

According to Lemma 5.4 and Remark 5.3, solutions of systems (7.1) and (1.3) are continuous with respect to initial values.

Since both $\tilde{H}(t, \cdot)$ and $\tilde{L}(t, \cdot)$ are continuous, $\tilde{H}(t, \cdot)$ and $\tilde{L}(t, \cdot)$ are homeomorphisms of \mathbb{R}^n . Thus System (1.5) is topologically conjugate to system (1.3). The proof is complete. \square

8 The proof of Theorem 2

From Lemmas 6.12 and 7.2, we have that $H(t, \cdot) \circ \tilde{H}(t, \cdot)$ and $\tilde{L}(t, \cdot) \circ L(t, \cdot)$ are homeomorphisms of \mathbb{R}^n and $(H(t, \cdot) \circ \tilde{H}(t, \cdot))^{-1} = \tilde{L}(t, \cdot) \circ L(t, \cdot)$. Moreover, $H(t, \cdot) \circ \tilde{H}(t, \cdot)$ sends the solutions of system (1.4) onto those of system (1.3) and $\tilde{L}(t, \cdot) \circ L(t, \cdot)$ sends the solutions of system (1.3) onto those of system (1.4). It is easy to see that $|H(t, \cdot) \circ \tilde{H}(t, (x, y)^T) - (x, y)^T|$ and $|\tilde{L}(t, \cdot) \circ L(t, (x, y)^T) - (x, y)^T|$ are bounded. Therefore system (1.3) and system (1.4) are topologically conjugated. \square

9 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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